Lecture No. 10

Measure and Integration

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(1)

let $A \subseteq \mathbb{R}$, A countable

$$A = \{x_1, x_2, \cdots\}$$

$$A = \bigcup_{i=1}^{n} \{x_i\}$$

And $\mu(\{x_i\}) = 0 \quad \forall i$

Thus $\mu^*(A) \leq \sum_{i=1}^{\infty} \mu(\{x_i\}) = 0$

$$\Longrightarrow \mu^*(A) = 0$$

if A is countable.

Let $A \subseteq \mathbb{R}$, A uncountable (2)

<u>Claim</u> $\mu^*(A) = 1$

<u>Note</u> $\mu^*(A) \leq 1 = \mu^*(X)$

<u>To show</u> $\mu^*(A) \geq 1$.

<u>Note</u> A uncountable, let

~~$\Rightarrow A$~~ $A \subseteq \bigcup_{i=1}^{\infty} A_i$, $A_i \in \mathcal{A}$

A uncountable $\Rightarrow$ at least one A_i is uncountable, say A_{i_0} is uncountable.

Thus $\mu(A_{i_0}) \neq 0$ ③

Because $A_{i_0} \in \mathcal{A}$, A_{i_0} uncountable

$\Rightarrow$ $A_{i_0}^c$ is finite

$\Rightarrow$ $\mu(A_{i_0}) = 1$

Thus $A \subseteq \bigcup_{i=1}^{\infty} A_i$, then for some i_0 $\mu(A_{i_0})=1$

$\Rightarrow$ $\mu^*(A) \geq 1$ $\left(\sum_{i=1}^{\infty} \mu(A_i) \geq 1\right)$

Hence $\mu^*(A) = 1$

Thus A uncountable $\Rightarrow$ $\mu^*(A)=1$

$\Leftarrow$ If $\mu^*(A)=1$ then A is uncountable.

$A \in \mathcal{A}$, then A is measurable:

$\forall \; Y \subseteq X$, $\mu^*(Y) < +\infty$,

$$\mu^*(Y) \geq \mu^*(Y \cap A) + \mu^*(Y \cap A^c)$$

Pf Let $\varepsilon > 0$ be fixed. Then

$\exists$ sets $A_i \in \mathcal{A}$, $Y \subseteq \bigcup_{i=1}^{\infty} A_i$ and

$$\mu^*(Y) + \varepsilon > \sum_{i=1}^{\infty} \mu(A_i) \quad \text{—— } (1)$$

$$\cancel{= \sum_{i=1}^{\infty} \mu\left(\left(\bigcup_{i=1}^{\infty} A_i \cap A\right) \cup \left(\bigcup_{i=1}^{\infty} A_i \cap A^c\right)\right)}$$

Note $A_i = (A_i \cap A) \cup (A_i \cap A^c)$

$$\Rightarrow \quad \mu(A_i) = \mu(A_i \cap A) + \mu(A_i \cap A^c)$$

$$\sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{\infty} \mu(A_i \cap A) + \sum_{i=1}^{\infty} \mu(A_i \cap A^c) \longrightarrow (2)$$

N.b

$$A \cap Y \subseteq \bigcup_{i=1}^{\infty} (A_i \cap A)$$

$$A^c \cap Y \subseteq \bigcup_{i=1}^{\infty} (A_i \cap A^c) \text{ —}$$

$$\Rightarrow \quad \mu^*(A \cap Y) \leq \sum_{i=1}^{\infty} \mu(A_i \cap A^c) \longrightarrow (3)$$

$$\mu^*(A^c \cap Y) \leq \sum_{i=1}^{\infty} \mu(A_i \cap A^c) \longrightarrow (4)$$

$$\mu^*(Y) + \varepsilon \geq \sum_{i=1}^{\infty} \mu^*(A_i)$$

$$\geq \sum_{i=1}^{\infty} \mu^*(A_i \cap A) + \sum_{i=1}^{\infty} \mu^*(A_i \cap A^c)$$

$$\geq \mu^*(Y \cap A) + \mu^*(Y \cap A^c)$$

ε is arbitrary. Let $\varepsilon \longrightarrow 0$

$$\Rightarrow \mu^*(Y) \geq \mu^*(Y \cap A) + \mu^*(Y \cap A^c)$$

$$\Rightarrow A \in \mathcal{S}^*$$

Hence $\mathcal{A} \subseteq \mathcal{S}^*$

S^* is an algebra

(i) $\mathcal{A} \subseteq S^* \Longrightarrow \underline{\phi, X} \in \mathcal{A} \underline{\subseteq S^*}$

(ii) $E \in S^* \Longrightarrow E^c \in S^*$

(iii) $E_1, E_2 \in S^* \Longrightarrow (E_1 \cup E_2) \in S^*$?

Pf <u>To check</u> $\forall Y \subseteq X$, $\mu^*(Y) < +\infty$

(?) $\mu^*(Y) = \mu^*(Y \cap (E_1 \cup E_2)) + \mu^*(Y \cap (E_1 \cup E_2)^c)$

<u>Note</u> $E_1 \in S^* \Rightarrow \forall Y$

$\mu^*(Y) = \mu^*(Y \cap E_1) + \mu^*(Y \cap E_1^c)$

$\Rightarrow \mu^*(Y \cap (E_1 \cup E_2)) = \mu^*(Y \cap E_1) + \mu^*(Y \cap (E_1 \cup E_2) \cap E_1^c$

(1) $\boxed{\mu^*(Y \cap (E_1 \cup E_2)) = \mu^*(Y \cap E_1) + \mu^*(Y \cap E_2 \cap E_1^c)}$ (12)

$$\mu^*(Y \cap (E_1 \cup E_2)^c) = \mu^*(Y \cap (E_1^c \cap E_2^c))$$

Since E_1 is measurable,

$$\mu^*(Y \cap E_2^c) = \mu^*(Y \cap E_1 \cap E_2^c) + \mu^*(Y \cap E_2^c \cap E_1^c)$$

$\Rightarrow$ (2) $\boxed{\mu^*(Y \cap E_1^c \cap E_2^c) = \mu^*(Y \cap E_2^c) - \mu^*(Y \cap E_1 \cap E_2^c)}$

Add (1) and (2)

$$\left.\begin{array}{l} \mu^*(Y \cap (E_1 \cup E_2)) \\ + \mu^*(Y \cap E_1^c \cap E_2^c) \end{array}\right\} = \begin{array}{l} \mu^*(Y \cap E_1) + \mu^*(Y \cap E_2 \cap E_1^c) \\ + \mu^*(Y \cap E_2^c) - \underline{\underline{\mu^*(Y \cap E_1 \cap E_2^c)}} \end{array}$$

E_2 measurable

$$\Rightarrow \mu^*(Y) = \mu^*(Y \cap E_2) + \mu^*(Y \cap E_2^c)$$

$$\Rightarrow \cancel{\mu^*(Y \cap E_2) =}$$

$$\mu^*(Y \cap E_1) = \mu^*(Y \cap E_1 \cap E_2) + \underline{\mu^*(Y \cap E_1 \cap E_2^c)}$$

$$-\underline{\mu^*(Y \cap E_1 \cap E_2^c)} = -\mu^*(Y \cap E_1) + \mu^*(Y \cap E_1 \cap E_2)$$

$$\{ \ \} = \cancel{\mu^*(Y \cap E_1)} + \underline{\mu^*(Y \cap E_2 \cap E_1^c)} \checkmark$$

$$+\mu^*(Y \cap E_2^c) - \cancel{\mu^*(Y \cap E)} + \underline{\mu^*(Y \cap E_1 \cap E_2)}$$

$$= \mu^*(Y \cap E_2^c) + \mu^*(Y \cap E_2)$$

$$= \mu^*(Y)$$

$$E_1, E_2 \in \mathcal{S}^* \implies E_1 \cup E_2 \in \mathcal{S}^*$$